

Occupation times of generalized Ornstein-Uhlenbeck processes with two-sided exponential jumps

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Abstract

For an Ornstein-Uhlenbeck process driven by a double exponential jump diffusion process, we obtain formulas for the joint Laplace transform of it and its occupation times. The approach used is remarkable and can be extended to investigate the occupation times of an Ornstein-Uhlenbeck process driven by a more general Lévy process.

Keywords: Ornstein-Uhlenbeck process; Occupation times; Exit problem.

1. Introduction

A generalized Ornstein-Uhlenbeck process $X = (X_t)_{t \geq 0}$ is characterized by the following equation:

$$dX_t = \kappa(\alpha - X_t)dt + dL_t, \quad t > 0, \quad (1.1)$$

where $\kappa > 0$, $\alpha \in \mathbb{R}$ and $X_0 = x$ is non-random; $L = (L_t)_{t \geq 0}$ is a Lévy process. The first passage time of X has been investigated considerably, the reader is referred to Jacobsen and Jensen (2007) and Borovkov and Novikov (2008) and literatures therein for the details.

In this article, we are interested in the joint Laplace transform of X and its occupation times, i.e., $E \left[e^{-p \int_0^T \mathbf{1}_{\{X_t \leq b\}} dt + q X_T} \right]$, where $p > 0$, q is some suitable constant and for a given set A , $\mathbf{1}_A$ is the indicator function; and the objection is deriving formulas for its Laplace transform, i.e.,

$$\int_0^\infty e^{-sT} E \left[e^{-p \int_0^T \mathbf{1}_{\{X_t \leq b\}} dt + q X_T} \right] dT, \quad \text{for } s > 0. \quad (1.2)$$

If L_t in (1.1) is a Brownian motion with drift, formulas for (1.2) with $q = 0$ are known, one can refer to Li and Zhou (2013) for example. Thus, we focus on the

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case that L_t is a jump diffusion process, and to the best of our knowledge, we are the first to investigate (1.2) under the assumption that L_t has jumps.

Results obtained here can be applied to price occupation time derivatives as in Cai et al. (2010), in which the authors have noted that there are several products in the real market with payoffs depending on the occupation times of an interest rate or a spread of swap rates (see Remark 3.3 in that paper). Usually, interest rates or spreads of swap rates are modeled by generalized Ornstein-Uhlenbeck processes since they are mean reversion. Therefore, our results are very important for pricing such derivatives.

The remainder of the paper is organized as follows. Section 2 presents the details of our model and some important preliminary results, and Section 3 derives the main results.

2. Details of the model and some preliminary outcomes

2.1. The model

In this paper, the Lévy process $L = (L_t)_{t \geq 0}$ in (1.1) is assumed to be a double exponential jump diffusion process, i.e.,

$$L_t = \mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_k. \quad (2.1)$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ are constants; $W = (W_t)_{t \geq 0}$ is a standard Brownian motion and is independent of $\sum_{k=1}^{N_t} Y_k$, which is a compound Poisson process with intensity λ and the following jump distribution:

$$f_Y(y) = P(Y_1 \in dy) / dy = p\eta e^{-\eta y} \mathbf{1}_{\{y > 0\}} + (1-p)\vartheta e^{\vartheta y} \mathbf{1}_{\{y < 0\}}, \quad (2.2)$$

where $0 < p < 1$ and $\eta, \vartheta > 0$. For the investigation on such a Lévy process L_t , we refer to Kou and Wang (2003).

In what follows, for given $x \in \mathbb{R}$, \mathbb{P}_x is the law of X starting from x with \mathbb{E}_x denoting the matching expectation. If $x = 0$, we drop the subscript and write simply \mathbb{P} and \mathbb{E} . Recall (1.1) and note that

$$X_t = \alpha + e^{-\kappa t}(X_0 - \alpha) + \int_0^t e^{\kappa(s-t)} dL_s, \quad t \geq 0,$$

and

$$\varphi(\theta) := \ln(\mathbb{E}[e^{\theta L_1}]) = \mu\theta + \frac{\sigma^2}{2}\theta^2 + \lambda \left(\frac{p\eta}{\eta - \theta} + \frac{(1-p)\vartheta}{\vartheta + \theta} - 1 \right), \quad -\vartheta < \theta < \eta.$$

It is known that (see Theorem 1 in Lukacs (1969))

$$\mathbb{E} \left[e^{\int_0^t \hat{f}(s) dL_s} \right] = e^{\int_0^t \varphi(\hat{f}(s)) ds},$$

where $\hat{f}(s)$ is a non-random function. Thus, for $-\vartheta < \theta < \eta$, we can derive

$$\begin{aligned} \ln(\mathbb{E}_x[e^{\theta X_t}]) &= \theta e^{-\kappa t} x + \theta \alpha (1 - e^{-\kappa t}) + \mu \theta \frac{1 - e^{-\kappa t}}{\kappa} \\ &+ \frac{\sigma^2(1 - e^{-2\kappa t})}{4\kappa} \theta^2 + \lambda \left(\frac{p}{\kappa} \ln\left(\frac{\eta - \theta e^{-\kappa t}}{\eta - \theta}\right) + \frac{(1-p)}{\kappa} \ln\left(\frac{\vartheta + \theta e^{-\kappa t}}{\vartheta + \theta}\right) \right), \end{aligned} \quad (2.3)$$

where in the above derivation, we have used the following identity:

$$t = \frac{p}{\kappa} \ln(e^{\kappa t}) + \frac{(1-p)}{\kappa} \ln(e^{\kappa t}).$$

For given $-\vartheta < \theta < \eta$, note that

$$\lim_{t \uparrow \infty} \ln(\mathbb{E}_x[e^{\theta X_t}]) = \theta \alpha + \frac{\mu \theta}{\kappa} + \frac{\sigma^2}{4\kappa} \theta^2 + \lambda \left(\frac{p}{\kappa} \ln\left(\frac{\eta}{\eta - \theta}\right) + \frac{(1-p)}{\kappa} \ln\left(\frac{\vartheta}{\vartheta + \theta}\right) \right). \quad (2.4)$$

For the process X given by (1.1) and (2.1), the purpose of the paper is to deduce formulas for (1.2), i.e.,

$$\int_0^\infty e^{-sT} \mathbb{E}_x \left[e^{-p \int_0^T \mathbf{1}_{\{X_t \leq b\}} dt + q X_T} \right] dT, \quad \text{for } s > 0. \quad (2.5)$$

Our approach depends on results about the one-sided exit problem of X , which will be presented in the next subsection.

Remark 2.1. In Cai et al. (2010), they have obtained expressions for (2.5) under the assumption that the process X is a double exponential jump diffusion process (i.e., $X_t = L_t$ or $\kappa = 0$ in (1.1)). A contribution here is extending their results to the case of $\kappa > 0$.

Remark 2.2. The method in this article can be extended to calculate (2.5) when X_t is given by (1.1) and L_t is a hyper-exponential jump diffusion process¹. But, to illustrate the ideas in our approach clearly, it is desirable to consider a simpler model.

2.2. Results on the one-sided exit problem of X

First of all, for $a, c \in \mathbb{R}$, define

$$\tau_a^- := \inf\{t > 0 : X_t \leq a\}, \quad \tau_c^+ := \inf\{t > 0 : X_t \geq c\}. \quad (2.6)$$

For the stopping time τ_a^- , $q > 0$ and $\xi \geq 0$, we want to compute the following quantities:

$$\mathbb{E}_x \left[e^{-q\tau_a^- - \xi(a - X_{\tau_a^-})} \mathbf{1}_{\{X_{\tau_a^-} < a\}} \right] \quad \text{and} \quad \mathbb{E}_x \left[e^{-q\tau_a^-} \mathbf{1}_{\{X_{\tau_a^-} = a\}} \right], \quad \text{for } x > a. \quad (2.7)$$

¹In other words, the distribution of Y_1 in (2.1) is generalized to the following form:

$$f_Y(y) = \sum_{i=1}^m p_i \eta_i e^{-\eta_i y} \mathbf{1}_{\{y > 0\}} + \sum_{j=1}^n q_j \vartheta_j e^{\vartheta_j y} \mathbf{1}_{\{y < 0\}}.$$

If $\sigma = \mu = \alpha = 0$ in (1.1) and (2.1), formulas for (2.7) have been derived in Jacobsen and Jensen (2007) (see Proposition 5), in which they also considered the case of $\sigma > 0$, $\alpha \in \mathbb{R}$ and $\mu = 0$. Applying similar ideas in Jacobsen and Jensen (2007) can lead to the following Lemma 2.1 for the case of $\mu \in \mathbb{R}$. Before giving Lemma 2.1, we introduce some notations.

For α given in (1.1), $q > 0$ and $x \in \mathbb{R}$, define

$$\hat{x} := x - \alpha \quad \text{and} \quad \psi_q(x) := x^{\frac{q}{\kappa}-1} e^{-\frac{q^2}{4\kappa}x^2 + \frac{\mu}{\kappa}x} (\eta + x)^{\frac{\lambda p}{\kappa}} (x - \vartheta)^{\frac{\lambda(1-p)}{\kappa}}, \quad (2.8)$$

where $\psi_q(x)$ is treated as a complex valued function when $x < \vartheta$. And for $\xi, \rho \geq 0$, $q > 0$, $x \in \mathbb{R}$ and $i = 1, 2, 3, 4$,

$$\begin{aligned} F_i^q(x) &:= \int_{\Gamma_i} |\psi_q(z)| e^{-xz} dz, \\ D_i^{\rho, q}(x) &:= \int_{\Gamma_i} \frac{\eta + \rho}{z + \eta} |\psi_q(z)| e^{-xz} dz, \\ C_i^{\xi, q}(x) &:= - \int_{\Gamma_i} \frac{\vartheta + \xi}{z - \vartheta} |\psi_q(z)| e^{-xz} dz, \end{aligned} \quad (2.9)$$

where $\Gamma_1 = (0, \vartheta)$, $\Gamma_2 = (\vartheta, \infty)$, $\Gamma_3 = (-\eta, 0)$ and $\Gamma_4 = (-\infty, -\eta)$; $|\psi_q(z)|$ is the module of $\psi_q(z)$.

Lemma 2.1. For $x > a$, $q > 0$ and $\xi \geq 0$,

$$\begin{pmatrix} \mathbb{E}_x \left[e^{-q\tau_a^-} \mathbf{1}_{\{X_{\tau_a^-} = a\}} \right] \\ \mathbb{E}_x \left[e^{-q\tau_a^- - \xi(a - X_{\tau_a^-})} \mathbf{1}_{\{X_{\tau_a^-} < a\}} \right] \end{pmatrix} = \begin{pmatrix} F_1^q(\hat{a}) & C_1^{\xi, q}(\hat{a}) \\ F_2^q(\hat{a}) & C_2^{\xi, q}(\hat{a}) \end{pmatrix}^{-1} \begin{pmatrix} F_1^q(\hat{x}) \\ F_2^q(\hat{x}) \end{pmatrix}. \quad (2.10)$$

Proof. The derivation depends on some similar ideas and calculations in the proof of Proposition 5 in Jacobsen and Jensen (2007).

(i) Assume $\alpha = 0$, then $x = \hat{x}$ (recall (2.8)) for any $x \in \mathbb{R}$. Consider a function $f_1(x)$ defined as follows:

$$f_1(x) = \begin{cases} F_1^q(x), & x \geq a, \\ C_1^{\xi, q}(a) e^{-\xi(a-x)}, & x < a. \end{cases} \quad (2.11)$$

Obviously, $f_1(x)$ is bounded and differentiable on $[a, \infty)$.

Let \mathcal{A} be the generator of X . In other words, for $x \geq a$,

$$\mathcal{A}f_1(x) = -\kappa x f_1'(x) + \mu f_1'(x) + \frac{\sigma^2}{2} f_1''(x) + \lambda \int_{-\infty}^{\infty} f_1(x+y) f_Y(y) dy - \lambda f_1(x),$$

where $f_Y(y)$ given by (2.2). Then, for $x \geq a$, some straightforward calculations

will lead to

$$\begin{aligned}
\mathcal{A}f_1(x) - qf_1(x) &= \kappa x \int_{\Gamma_1} z|\psi_q(z)|e^{-xz}dz - \mu \int_{\Gamma_1} z|\psi_q(z)|e^{-xz}dz \\
&+ \frac{\sigma^2}{2} \int_{\Gamma_1} z^2|\psi_q(z)|e^{-xz}dz + \lambda p \int_{\Gamma_1} \frac{\eta}{\eta+z} |\psi_q(z)|e^{-xz}dz \\
&+ \lambda(1-p)\vartheta \int_{\Gamma_1} \frac{1 - e^{(\vartheta-z)(a-x)}}{\vartheta-z} |\psi_q(z)|e^{-xz}dz - \lambda p f_1(x) \\
&- \lambda(1-p)f_1(x) - qf_1(x) + C_1^{\xi,q}(a)\lambda(1-p)\frac{\vartheta}{\vartheta+\xi}e^{\vartheta(a-x)},
\end{aligned} \tag{2.12}$$

where the fourth and fifth integral on the right-hand side of (2.12) follows from exchanging the order of integration.

Recall $\Gamma_1 = (0, \vartheta)$. For the first integral on the right-hand side of (2.12), applying partial integration yields

$$\kappa x \int_0^\vartheta z|\psi_q(z)|e^{-xz}dz = \kappa \int_0^\vartheta \left(\frac{\partial|\psi_q(z)|}{\partial z} z + |\psi_q(z)| \right) e^{-xz}dz, \tag{2.13}$$

where we have used that

$$\lim_{z \downarrow 0} z|\psi_q(z)| = 0 \quad \text{and} \quad \lim_{z \uparrow \vartheta} z|\psi_q(z)| = 0.$$

From (2.9), (2.11) and (2.13), we can write the right-hand side of (2.12) as

$$\int_{\Gamma_1} e^{-xz} \left(\kappa \frac{\partial|\psi_q(z)|}{\partial z} z + |\psi_q(z)| \left(\kappa + \frac{\sigma^2}{2} z^2 - \mu z - \frac{\lambda p z}{\eta+z} + \frac{\lambda(1-p)z}{\vartheta-z} - q \right) \right) dz,$$

which combined with the definition of $\psi_q(z)$ in (2.8), produces²

$$\mathcal{A}f_1(x) - qf_1(x) = 0, \quad \text{for } x \geq a. \tag{2.14}$$

From (2.14), Itô's formula and the dominated convergence theorem will give

$$\lim_{t \uparrow \infty} \mathbb{E}_x \left[e^{-q(\tau_a^- \wedge t)} f_1(X_{\tau_a^- \wedge t}) \right] = \mathbb{E}_x \left[e^{-q\tau_a^-} f_1(X_{\tau_a^-}) \right] = f_1(x), \quad x \geq a. \tag{2.15}$$

It follows from (2.15) and the definition of $f_1(x)$ for $x < a$ in (2.11) that

$$C_1^{\xi,q}(a) \mathbb{E}_x \left[e^{-q\tau_a^- - \xi(a - X_{\tau_a^-})} \mathbf{1}_{\{X_{\tau_a^-} < a\}} \right] + \mathbb{E}_x \left[e^{-q\tau_a^-} \mathbf{1}_{\{X_{\tau_a^-} = a\}} \right] f_1(a) = f_1(x), \quad x \geq a. \tag{2.16}$$

²In fact, the expression of $|\psi_q(z)|$ for $0 < z < \vartheta$ is obtained by solving the equation:

$$\kappa \frac{\partial|\psi_q(z)|}{\partial z} z + |\psi_q(z)| \left(\kappa + \frac{\sigma^2}{2} z^2 - \mu z - \lambda p \frac{z}{\eta+z} + \lambda(1-p) \frac{z}{\vartheta-z} - q \right) = 0.$$

Similarly, define

$$f_2(x) = \begin{cases} F_2^q(x), & x \geq a, \\ C_2^{\xi,q}(a)e^{-\xi(a-x)}, & x < a, \end{cases} \quad (2.17)$$

where $F_2^q(x)$ and $C_2^{\xi,q}(a)$ are given by (2.9). Note that $\Gamma_2 = (\vartheta, \infty)$ and $|\psi_q(z)| = \psi_q(z)$ for $z > \vartheta$. In addition,

$$\lim_{z \downarrow \vartheta} z\psi_q(z) = 0 \quad \text{and} \quad \lim_{z \uparrow \infty} z\psi_q(z) = 0,$$

and for $z > \vartheta$,

$$\kappa\psi_q'(z)z + \psi_q(z) \left(\kappa + \frac{\sigma^2}{2}z^2 - \mu z - \lambda p \frac{z}{\eta + z} + \lambda(1-p) \frac{z}{\vartheta - z} - q \right) = 0.$$

Thus, for $x \geq a$, it can be proved that

$$C_2^{\xi,q}(a)\mathbb{E}_x \left[e^{-q\tau_a^- - \xi(a - X_{\tau_a^-})} \mathbf{1}_{\{X_{\tau_a^-} < a\}} \right] + \mathbb{E}_x \left[e^{-q\tau_a^-} \mathbf{1}_{\{X_{\tau_a^-} = a\}} \right] f_2(a) = f_2(x). \quad (2.18)$$

Therefore, formula (2.10) for $\alpha = 0$ is derived from (2.11) and (2.16)–(2.18).

(ii) If $\alpha \neq 0$, let $\hat{X}_t = X_t - \alpha$ for $t \geq 0$ and note that

$$d\hat{X}_t = -\kappa\hat{X}_t dt + dL_t.$$

This yields the desired result. \square

Remark 2.3. Lemma 2.1 implies that

$$\mathbb{E}_x \left[e^{-q\tau_a^- - \xi(a - X_{\tau_a^-})} \mathbf{1}_{\{X_{\tau_a^-} < a\}} \right] = \frac{F_1^q(\hat{a})F_2^q(\hat{x}) - F_2^q(\hat{a})F_1^q(\hat{x})}{C_2^{\xi,q}(\hat{a})F_1^q(\hat{a}) - C_1^{\xi,q}(\hat{a})F_2^q(\hat{a})}. \quad (2.19)$$

Recall (2.9) and note that the right-hand side of (2.19) can be written as $\hat{F}(\hat{a}, \hat{x})/(\vartheta + \xi)$, where $\hat{F}(\hat{a}, \hat{x})$ is not dependent on ξ . So formula (2.19) confirms the following well-known result:

$$\mathbb{E}_x \left[e^{-q\tau_a^-} \mathbf{1}_{\{a - X_{\tau_a^-} \in dy\}} \right] = \mathbb{E}_x \left[e^{-q\tau_a^-} \mathbf{1}_{\{X_{\tau_a^-} < a\}} \right] \vartheta e^{-\vartheta y} dy, \quad \text{for } y > 0,$$

which is due to the lack of memory of exponential distributions

For the stopping time τ_c^+ in (2.6), similar results to Lemma 2.1 hold.

Lemma 2.2. For $x < c$, $q > 0$ and $\rho \geq 0$,

$$\begin{pmatrix} \mathbb{E}_x \left[e^{-q\tau_c^+} \mathbf{1}_{\{X_{\tau_c^+} = c\}} \right] \\ \mathbb{E}_x \left[e^{-q\tau_c^+ - \rho(X_{\tau_c^+} - c)} \mathbf{1}_{\{X_{\tau_c^+} > c\}} \right] \end{pmatrix} = \begin{pmatrix} F_3^q(\hat{c}) & D_3^{\rho,q}(\hat{c}) \\ F_4^q(\hat{c}) & D_4^{\rho,q}(\hat{c}) \end{pmatrix}^{-1} \begin{pmatrix} F_3^q(\hat{x}) \\ F_4^q(\hat{x}) \end{pmatrix}. \quad (2.20)$$

Proof. From the derivation of Lemma 2.1, it is enough to consider the case of $\alpha = 0$. Thus we assume that $x = \hat{x}$ in this proof. For $i = 1, 2$, consider the following function

$$g_i(x) = \begin{cases} F_{2+i}^q(x), & x \leq c, \\ D_{2+i}^{\rho,q}(c)e^{-\rho(x-c)}, & x > c, \end{cases} \quad (2.21)$$

where $F_{2+i}^q(x)$ and $D_{2+i}^{\rho,q}(c)$ are given by (2.9). Similar to the derivation of (2.15), it can be shown that

$$\mathbb{E}_x \left[e^{-q\tau_c^+} g_i(X_{\tau_c^+}) \right] = g_i(x), \quad \text{for } x \leq c. \quad (2.22)$$

This result and the definition of $g_i(x)$ for $x > c$ in (2.21) give us

$$D_{2+i}^{\rho,q}(c) \mathbb{E}_x \left[e^{-q\tau_c^+ - \rho(X_{\tau_c^+} - c)} \mathbf{1}_{\{X_{\tau_c^+} > c\}} \right] + \mathbb{E}_x \left[e^{-q\tau_c^+} \mathbf{1}_{\{X_{\tau_c^+} = c\}} \right] g_i(c) = g_i(x), \quad x \leq c,$$

from which (2.20) is deduced. \square

Similar to Remark 2.3, it also holds that

$$\mathbb{E}_x \left[e^{-q\tau_c^+} \mathbf{1}_{\{X_{\tau_c^+} - c \in dy\}} \right] = \mathbb{E}_x \left[e^{-q\tau_c^+} \mathbf{1}_{\{X_{\tau_c^+} > c\}} \right] \eta e^{-\eta y} dy, \quad \text{for } y > 0.$$

In particular, we have the following lemma.

Lemma 2.3. (i) For any nonnegative measurable function $f(x)$ on \mathbb{R} such that $\int_{-\infty}^0 f(a+y)e^{\vartheta y} dy < \infty$, $q > 0$ and $x > a$, it holds that

$$\begin{aligned} \mathbb{E}_x \left[e^{-q\tau_a^-} f(X_{\tau_a^-}) \right] &= \mathbb{E}_x \left[e^{-q\tau_a^-} \mathbf{1}_{\{X_{\tau_a^-} < a\}} \right] \int_{-\infty}^0 f(a+y) \vartheta e^{\vartheta y} dy \\ &\quad + f(a) \mathbb{E}_x \left[e^{-q\tau_a^-} \mathbf{1}_{\{X_{\tau_a^-} = a\}} \right]. \end{aligned} \quad (2.23)$$

(ii) For any nonnegative measurable function $f(x)$ on \mathbb{R} such that $\int_0^\infty f(c+y)e^{-\eta y} dy < \infty$, $q > 0$ and $x < c$, it holds that

$$\begin{aligned} \mathbb{E}_x \left[e^{-q\tau_c^+} f(X_{\tau_c^+}) \right] &= \mathbb{E}_x \left[e^{-q\tau_c^+} \mathbf{1}_{\{X_{\tau_c^+} > c\}} \right] \int_0^\infty f(c+y) \eta e^{-\eta y} dy \\ &\quad + f(c) \mathbb{E}_x \left[e^{-q\tau_c^+} \mathbf{1}_{\{X_{\tau_c^+} = c\}} \right]. \end{aligned} \quad (2.24)$$

3. Main results

In this section, assume that $b \in \mathbb{R}$, $s > 0$, $p > -s$ and $-\vartheta < 3 \cdot q < \eta$. The objection is to deduce the expression of

$$\begin{aligned} V(x) &:= \int_0^\infty s e^{-sT} \mathbb{E}_x \left[e^{-p \int_0^T \mathbf{1}_{\{X_t \leq b\}} dt + q X_T} \right] dT \\ &= \mathbb{E}_x \left[e^{-p \int_0^{\epsilon(s)} \mathbf{1}_{\{X_t \leq b\}} dt + q X_{\epsilon(s)}} \right], \end{aligned} \quad (3.1)$$

where the variable $e(s)$, independent of X , is an exponential distribution with parameter s . Recall (2.3) and (2.4). Since $-\vartheta < 3 \cdot q < \eta$, we have

$$\mathbb{E}_x [e^{qX_{e(s)}}] = s \int_0^\infty e^{-st} \mathbb{E}_x [e^{qX_t}] dt < \infty,$$

for any given $s > 0$ and $x \in \mathbb{R}$. For $x \in \mathbb{R}$, define

$$\begin{aligned} V_1^s(x) &:= \mathbb{E}_x [e^{qX_{e(s)}}], \\ T_\eta^s(x) &= \int_0^\infty V_1^s(x+z) \eta e^{-\eta z} dz, \\ T_\vartheta^s(x) &= \int_{-\infty}^0 V_1^s(x+z) \vartheta e^{\vartheta z} dz. \end{aligned} \quad (3.2)$$

Due to (2.3), $V_1^s(x)$, $T_\eta^s(x)$ and $T_\vartheta^s(x)$ are considered as known functions from now on.

The main results are given in Theorem 3.1, and for its derivation, we improve the approach in Wu and Zhou (2016). Especially, the technic used in proving $V'(b-) = V'(b+)$ in Lemma 3.1 (will be presented after the proof of Theorem 3.1) is new and novel, and is expected to give some motivations to the investigation on the occupation times of Ornstein-Uhlenbeck processes driven by more general Lévy processes and other stochastic processes.

Theorem 3.1. *For $s > 0$, $p > -s$, $b \in \mathbb{R}$ and $-\vartheta < 3 \cdot q < \eta$, we have*

$$\mathbb{E}_x \left[e^{-p \int_0^{e(s)} \mathbf{1}_{\{X_t \leq b\}} dt + qX_{e(s)}} \right] = \begin{cases} \frac{s}{k} V_1^k(x) + \sum_{i=1}^2 J_i F_{2+i}^k(\hat{x}), & x \leq b, \\ V_1^s(x) + \sum_{i=1}^2 N_i F_i^s(\hat{x}), & x \geq b, \end{cases} \quad (3.3)$$

where $k = s + p$ and the constants J_i and N_i satisfy

$$(J_1, J_2, N_1, N_2) \mathbb{Q} = w, \quad (3.4)$$

with

$$w = \left(V_1^s(b) - \frac{sV_1^k(b)}{k}, V_1^{s'}(b) - \frac{sV_1^{k'}(b)}{k}, T_\eta^s(b) - \frac{s}{k} T_\eta^k(b), T_\vartheta^s(b) - \frac{s}{k} T_\vartheta^k(b) \right),$$

and

$$\mathbb{Q} = \begin{pmatrix} F_3^k(\hat{b}) & F_3^{k'}(\hat{b}) & D_3^{0,k}(\hat{b}) & C_3^{0,k}(\hat{b}) \\ F_4^k(\hat{b}) & F_4^{k'}(\hat{b}) & D_4^{0,k}(\hat{b}) & C_4^{0,k}(\hat{b}) \\ -F_1^s(\hat{b}) & -F_1^{s'}(\hat{b}) & -D_1^{0,s}(\hat{b}) & -C_1^{0,s}(\hat{b}) \\ -F_2^s(\hat{b}) & -F_2^{s'}(\hat{b}) & -D_2^{0,s}(\hat{b}) & -C_2^{0,s}(\hat{b}) \end{pmatrix}. \quad (3.5)$$

Here, in (3.3) and (3.5), for any given $r > 0$, $F_i^r(\hat{b})$, $D_i^{0,r}(\hat{b})$ and $C_i^{0,r}(\hat{b})$ are given by (2.9).

Proof. In this derivation, some similar ideas in Wu and Zhou (2016) will be used. First, we know from (3.1) that

$$V(x) \leq \begin{cases} \mathbb{E}_x [e^{qX_{e(s)}}], & \text{if } p \geq 0, \\ \mathbb{E}_x [e^{-pe(s) + qX_{e(s)}}], & \text{if } -s < p < 0, \end{cases} \quad (3.6)$$

and note that

$$\mathbb{E}_x \left[e^{-pe(s)+qX_{e(s)}} \right] = s \int_0^\infty e^{-st} \mathbb{E}_x \left[e^{-pt+qX_t} \right] dt = \frac{s}{s+p} \mathbb{E}_x \left[e^{qX_{e(p+s)}} \right]. \quad (3.7)$$

For $x < b$, it follows from the lack of memory property of $e(s)$ and the strong Markov property of X that (recall (3.1))

$$\begin{aligned} V(x) &= \mathbb{E}_x \left[e^{-p\tau_b^+ - p \int_{\tau_b^+}^{e(s)} \mathbf{1}_{\{X_t \leq b\}} dt + qX_{e(s)}} \mathbf{1}_{\{e(s) > \tau_b^+\}} \right] \\ &\quad + \mathbb{E}_x \left[e^{-pe(s)+qX_{e(s)}} \mathbf{1}_{\{e(s) \leq \tau_b^+\}} \right] \\ &= \mathbb{E}_x \left[e^{-\kappa\tau_b^+} V(X_{\tau_b^+}) \right] + \frac{s}{\kappa} \mathbb{E}_x \left[e^{qX_{e(\kappa)}} \mathbf{1}_{\{e(\kappa) \leq \tau_b^+\}} \right] \\ &= \mathbb{E}_x \left[e^{-\kappa\tau_b^+} \left(V(X_{\tau_b^+}) - \frac{s}{\kappa} V_1^\kappa(X_{\tau_b^+}) \right) \right] + \frac{s}{\kappa} V_1^\kappa(x) \\ &= \frac{s}{\kappa} V_1^\kappa(x) + \sum_{i=1}^2 J_i F_{2+i}^\kappa(\hat{x}), \end{aligned} \quad (3.8)$$

where $k = s + p$ and $V_1^k(x)$ is given by (3.2); the final equality follows from (2.20), (2.24) and (3.2) with

$$(J_1, J_2) = \left(V(b) - \frac{s}{k} V_1^k(b), \int_0^\infty V(b+z) \eta e^{-\eta z} dz - \frac{s}{k} T_\eta^k(b) \right) \begin{pmatrix} F_3^k(\hat{b}) & D_3^{0,k}(\hat{b}) \\ F_4^k(\hat{b}) & D_4^{0,k}(\hat{b}) \end{pmatrix}^{-1}. \quad (3.9)$$

Similarly, for $x > b$, formulas (2.10), (2.23) and (3.2) lead to

$$\begin{aligned} V(x) &= \mathbb{E}_x \left[e^{-p \int_0^{e(s)} \mathbf{1}_{\{X_t \leq b\}} dt + qX_{e(s)}} \mathbf{1}_{\{e(s) \leq \tau_b^-\}} \right] \\ &\quad + \mathbb{E}_x \left[e^{-p \int_{\tau_b^-}^{e(s)} \mathbf{1}_{\{X_t \leq b\}} dt + qX_{e(s)}} \mathbf{1}_{\{e(s) > \tau_b^-\}} \right] \\ &= \mathbb{E}_x \left[e^{qX_{e(s)}} \mathbf{1}_{\{e(s) \leq \tau_b^-\}} \right] + \mathbb{E}_x \left[e^{-s\tau_b^-} V(X_{\tau_b^-}) \right] \\ &= \mathbb{E}_x \left[e^{qX_{e(s)}} \right] - \mathbb{E}_x \left[e^{qX_{e(s)}} \mathbf{1}_{\{e(s) > \tau_b^-\}} \right] + \mathbb{E}_x \left[e^{-s\tau_b^-} V(X_{\tau_b^-}) \right] \\ &= V_1^s(x) + \mathbb{E}_x \left[e^{-s\tau_b^-} (V(X_{\tau_b^-}) - V_1^s(X_{\tau_b^-})) \right] = V_1^s(x) + \sum_{i=1}^2 N_i F_i^s(\hat{x}), \end{aligned} \quad (3.10)$$

where

$$(N_1, N_2) \begin{pmatrix} F_1^s(\hat{b}) & C_1^{0,s}(\hat{b}) \\ F_2^s(\hat{b}) & C_2^{0,s}(\hat{b}) \end{pmatrix} = \left(V(b) - V_1^s(b), \int_{-\infty}^0 V(b+z) \vartheta e^{\vartheta z} dz - T_\vartheta^s(b) \right). \quad (3.11)$$

Formulas (3.9) and (3.11) imply

$$V(b) = \frac{s}{k} V_1^k(b) + \sum_{i=1}^2 J_i F_{2+i}^k(\hat{b}) = \sum_{i=1}^2 N_i F_i^s(\hat{b}) + V_1^s(b). \quad (3.12)$$

Besides, we know $V'(b-) = V'(b+)$ (see Lemma 3.1), which combined with (3.8) and (3.10), leads to

$$\sum_{i=1}^2 J_i \frac{\partial}{\partial \hat{x}} \left(F_{2+i}^k(\hat{x}) \right)_{\hat{x}=\hat{b}} + \frac{s}{k} V_1^{k'}(b) = \sum_{i=1}^2 N_i \frac{\partial}{\partial \hat{x}} \left(F_i^s(\hat{x}) \right)_{\hat{x}=\hat{b}} + V_1^{s'}(b). \quad (3.13)$$

From (2.9), (3.2) and (3.10), we can derive

$$\begin{aligned} \int_0^\infty V(b+z) \eta e^{-\eta z} dz &= \int_0^\infty V_1^s(b+z) \eta e^{-\eta z} dz + \sum_{i=1}^2 N_i \int_0^\infty F_i^s(\hat{b}+z) \eta e^{-\eta z} dz \\ &= T_\eta^s(b) + \sum_{i=1}^2 N_i D_i^{0,s}(\hat{b}), \end{aligned}$$

this result and formula (3.9) mean

$$\sum_{i=1}^2 J_i D_{2+i}^{0,k}(\hat{b}) + \frac{s}{k} T_\eta^k(b) = T_\eta^s(b) + \sum_{i=1}^2 N_i D_i^{0,s}(\hat{b}). \quad (3.14)$$

Applying similar derivations to (3.8) and (3.11) and using (3.2), we have

$$\begin{aligned} \sum_{i=1}^2 N_i C_i^{0,s}(\hat{b}) + T_\vartheta^s(b) &= \int_{-\infty}^0 V(b+z) \vartheta e^{\vartheta z} dz \\ &= \sum_{j=1}^2 J_j \int_{-\infty}^0 F_{2+j}^k(\hat{b}+z) \vartheta e^{\vartheta z} dz + \frac{s}{k} T_\vartheta^k(b) = \sum_{j=1}^2 J_j C_{2+j}^{0,k}(\hat{b}) + \frac{s}{k} T_\vartheta^k(b), \end{aligned} \quad (3.15)$$

where the last equality is due to (2.9). So, (3.3) is derived from (3.12)–(3.15) and the proof is completed. \square

Lemma 3.1. *For the function $V(x)$, defined in (3.1), its derivative is continuous at b , i.e., $V'(b-) = V'(b+)$.*

Proof. (i) Define the continuous component of X as X^c , i.e., $X_t^c = \alpha + e^{-\kappa t}(X_0 - \alpha) + \int_0^t e^{\kappa(s-t)} dL_s^c$, where $L_t^c = \mu t + \sigma W_t$ for $t \geq 0$, and introduce the stopping time $\tau_{b,\varepsilon}^c := \inf\{t > 0 : X_t^c > b + \varepsilon \text{ or } X_t^c < b - \varepsilon\}$ for $\varepsilon > 0$.

Expressions of $\mathbb{E}_b \left[e^{-q\tau_{b,\varepsilon}^c} \mathbf{1}_{\{X_{\tau_{b,\varepsilon}^c}^c = b+\varepsilon\}} \right]$ and $\mathbb{E}_b \left[e^{-q\tau_{b,\varepsilon}^c} \mathbf{1}_{\{X_{\tau_{b,\varepsilon}^c}^c = b-\varepsilon\}} \right]$ for $q > 0$ are known, one can refer to Borodin and Salminen (2002). Actually, for $\alpha = 0$, applying a similar but simple discussion to Lemma 2.1, we can obtain

$$\mathbb{E}_x \left[e^{-q\tau_{b,\varepsilon}^c} L_i(X_{\tau_{b,\varepsilon}^c}^c) \right] = L_i(x), \quad b - \varepsilon \leq x \leq b + \varepsilon, \quad (3.16)$$

where for $i = 1, 2$,

$$L_i(x) = \int_{\Pi_i} |z|^{\frac{a}{\kappa}-1} e^{-\frac{\sigma^2}{4\kappa} z^2 + \frac{\mu}{\kappa} z} e^{-xz} dz, \quad b - \varepsilon \leq x \leq b + \varepsilon,$$

with $\Pi_1 = (0, \infty)$ and $\Pi_2 = (-\infty, 0)$. Formula (3.16) is enough to obtain the desired result for $\alpha = 0$, and the case of $\alpha \in \mathbb{R}$ can be treated similarly as in Lemma 2.1. In short, it holds that

$$\mathbb{E}_b \left[e^{-q\tau_{b,\varepsilon}^c} \mathbf{1}_{\{X_{\tau_{b,\varepsilon}^c}^c = b+\varepsilon\}} \right] = \frac{L_1(\hat{b})L_2(\hat{b}-\varepsilon) - L_2(\hat{b})L_1(\hat{b}-\varepsilon)}{L_1(\hat{b}+\varepsilon)L_2(\hat{b}-\varepsilon) - L_2(\hat{b}+\varepsilon)L_1(\hat{b}-\varepsilon)}, \quad (3.17)$$

and

$$\mathbb{E}_b \left[e^{-q\tau_{b,\varepsilon}^c} \mathbf{1}_{\{X_{\tau_{b,\varepsilon}^c}^c = b-\varepsilon\}} \right] = \frac{L_2(\hat{b})L_1(\hat{b}+\varepsilon) - L_1(\hat{b})L_2(\hat{b}+\varepsilon)}{L_1(\hat{b}+\varepsilon)L_2(\hat{b}-\varepsilon) - L_2(\hat{b}+\varepsilon)L_1(\hat{b}-\varepsilon)}. \quad (3.18)$$

For given $q > 0$, after some straightforward calculations, we arrive at

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \frac{\frac{1}{2} - \mathbb{E}_b \left[e^{-q\tau_{b,\varepsilon}^c} \mathbf{1}_{\{X_{\tau_{b,\varepsilon}^c}^c = b-\varepsilon\}} \right]}{\varepsilon} &= \frac{L_1''(\hat{b})L_2(\hat{b}) - L_2''(\hat{b})L_1(\hat{b})}{4 \left(L_1(\hat{b})L_2'(\hat{b}) - L_2(\hat{b})L_1'(\hat{b}) \right)}, \\ \lim_{\varepsilon \downarrow 0} \frac{\frac{1}{2} - \mathbb{E}_b \left[e^{-q\tau_{b,\varepsilon}^c} \mathbf{1}_{\{X_{\tau_{b,\varepsilon}^c}^c = b+\varepsilon\}} \right]}{\varepsilon} &= -\frac{L_1''(\hat{b})L_2(\hat{b}) - L_2''(\hat{b})L_1(\hat{b})}{4 \left(L_1(\hat{b})L_2'(\hat{b}) - L_2(\hat{b})L_1'(\hat{b}) \right)}, \end{aligned} \quad (3.19)$$

and

$$\lim_{\varepsilon \downarrow 0} \frac{1 - \mathbb{E}_b \left[e^{-q\tau_{b,\varepsilon}^c} \right]}{\varepsilon^2} = \frac{L_1''(\hat{b})L_2'(\hat{b}) - L_1'(\hat{b})L_2''(\hat{b})}{2 \left(L_1(\hat{b})L_2'(\hat{b}) - L_2(\hat{b})L_1'(\hat{b}) \right)}. \quad (3.20)$$

Note that $L_1'(\hat{b}) < 0$ and $L_2'(\hat{b}) > 0$, thus $L_1(\hat{b})L_2'(\hat{b}) - L_2(\hat{b})L_1'(\hat{b}) \neq 0$.

(ii) It is known from (3.8) and (3.10) that both $V'(b-)$ and $V'(b+)$ exist, so it is enough to establish the following identity:

$$\lim_{\varepsilon \downarrow 0} \frac{(V(b+\varepsilon) + V(b-\varepsilon))/2 - V(b)}{\varepsilon} = 0. \quad (3.21)$$

By recalling (3.1) and letting T_1 denote the first jump time of the Poisson process N_t in (2.1), we deduce

$$\begin{aligned} V(b) &= \mathbb{E}_b \left[e^{-p \int_0^{e(s)} \mathbf{1}_{\{X_t \leq b\}} dt + q X_{e(s)}} \right] \\ &= \mathbb{E}_b \left[e^{-p \int_0^{e(s)} \mathbf{1}_{\{X_t \leq b\}} dt + q X_{e(s)}} \mathbf{1}_{\{e(s) > \tau_{b,\varepsilon}^c\}} \mathbf{1}_{\{T_1 \leq \tau_{b,\varepsilon}^c\}} \right] \\ &\quad + \mathbb{E}_b \left[e^{-p \int_0^{e(s)} \mathbf{1}_{\{X_t \leq b\}} dt + q X_{e(s)}} \mathbf{1}_{\{e(s) > \tau_{b,\varepsilon}^c\}} \mathbf{1}_{\{T_1 > \tau_{b,\varepsilon}^c\}} \right] \\ &\quad + \mathbb{E}_b \left[e^{-p \int_0^{e(s)} \mathbf{1}_{\{X_t \leq b\}} dt + q X_{e(s)}} \mathbf{1}_{\{e(s) \leq \tau_{b,\varepsilon}^c\}} \right]. \end{aligned} \quad (3.22)$$

For any $s > 0$, recall that $e(s)$ is independent of $\tau_{b,\varepsilon}^c$ and note that

$$\begin{aligned} 0 &\leq \lim_{\varepsilon \downarrow 0} \mathbb{E}_b \left[e^{q X_{e(s)}} \mathbf{1}_{\{e(s) \leq \tau_{b,\varepsilon}^c\}} \right] / \varepsilon \\ &\leq \lim_{\varepsilon \downarrow 0} \left(\mathbb{E}_b \left[e^{3q X_{e(s)}} \right] \right)^{\frac{1}{3}} \left(\mathbb{E}_b \left[\mathbf{1}_{\{e(s) \leq \tau_{b,\varepsilon}^c\}} \right] / \varepsilon^{3/2} \right)^{\frac{2}{3}} = 0, \end{aligned} \quad (3.23)$$

where the equality follows from (3.20).

For the first term on the right-hand side of (3.22), we have

$$\begin{aligned} & \mathbb{E}_b \left[e^{-p \int_0^{e(s)} \mathbf{1}_{\{X_t \leq b\}} dt + q X_{e(s)} \mathbf{1}_{\{e(s) > \tau_{b,\varepsilon}^c\}} \mathbf{1}_{\{T_1 \leq \tau_{b,\varepsilon}^c\}}} \right] \\ & \leq \begin{cases} \mathbb{E}_b \left[e^{q X_{e(s)} \mathbf{1}_{\{T_1 \leq \tau_{b,\varepsilon}^c\}}} \right], & \text{if } p \geq 0, \\ \frac{s}{s+p} \mathbb{E}_b \left[e^{q X_{e(s+p)} \mathbf{1}_{\{T_1 \leq \tau_{b,\varepsilon}^c\}}} \right], & \text{if } -s < p < 0. \end{cases} \end{aligned} \quad (3.24)$$

From (3.23) and (3.24), we arrive at (since T_1 is an exponentially distributed random and independent of $\tau_{b,\varepsilon}^c$)

$$0 \leq \lim_{\varepsilon \downarrow 0} \mathbb{E}_b \left[e^{-p \int_0^{e(s)} \mathbf{1}_{\{X_t \leq b\}} dt + q X_{e(s)} \mathbf{1}_{\{e(s) > \tau_{b,\varepsilon}^c\}} \mathbf{1}_{\{T_1 \leq \tau_{b,\varepsilon}^c\}}} \right] / \varepsilon = 0. \quad (3.25)$$

Similar calculations show that

$$0 \leq \lim_{\varepsilon \downarrow 0} \mathbb{E}_b \left[e^{-p \int_0^{e(s)} \mathbf{1}_{\{X_t \leq b\}} dt + q X_{e(s)} \mathbf{1}_{\{e(s) \leq \tau_{b,\varepsilon}^c\}}} \right] / \varepsilon \leq 0. \quad (3.26)$$

Note that $\{X_t, t < T_1\}$ and $\{X_t^c, t < T_1\}$ have the same distribution. This fact and the application of the strong Markov property of X will yield

$$\begin{aligned} & \mathbb{E}_b \left[e^{-p \int_0^{e(s)} \mathbf{1}_{\{X_t \leq b\}} dt + q X_{e(s)} \mathbf{1}_{\{e(s) > \tau_{b,\varepsilon}^c\}} \mathbf{1}_{\{T_1 > \tau_{b,\varepsilon}^c\}}} \right] \\ & = \mathbb{E}_b \left[e^{-p \int_0^{\tau_{b,\varepsilon}^c} \mathbf{1}_{\{X_t^c \leq b\}} dt} e^{-s \tau_{b,\varepsilon}^c} \mathbf{1}_{\{T_1 > \tau_{b,\varepsilon}^c\}} V(X_{\tau_{b,\varepsilon}^c}^c) \right] \\ & = \mathbb{E}_b \left[e^{-p \int_0^{\tau_{b,\varepsilon}^c} \mathbf{1}_{\{X_t^c \leq b\}} dt} e^{-(s+\lambda) \tau_{b,\varepsilon}^c} \mathbf{1}_{\{X_{\tau_{b,\varepsilon}^c}^c = b+\varepsilon\}} \right] V(b+\varepsilon) \\ & \quad + \mathbb{E}_b \left[e^{-p \int_0^{\tau_{b,\varepsilon}^c} \mathbf{1}_{\{X_t^c \leq b\}} dt} e^{-(s+\lambda) \tau_{b,\varepsilon}^c} \mathbf{1}_{\{X_{\tau_{b,\varepsilon}^c}^c = b-\varepsilon\}} \right] V(b-\varepsilon), \end{aligned} \quad (3.27)$$

where the second equality is due to the fact that T_1 is independent of X^c and $\tau_{b,\varepsilon}^c$. For the sake of brevity, the two items on the right-hand side of (3.27) are denoted respectively by $T_{b+\varepsilon} \cdot V(b+\varepsilon)$ and $T_{b-\varepsilon} \cdot V(b-\varepsilon)$. It is clear that

$$e^{-(s+\lambda+\max\{p,0\})\tau_{b,\varepsilon}^c} \leq e^{-p \int_0^{\tau_{b,\varepsilon}^c} \mathbf{1}_{\{X_s^c \leq b\}} ds} e^{-(s+\lambda)\tau_{b,\varepsilon}^c} \leq e^{-(s+\lambda+\min\{p,0\})\tau_{b,\varepsilon}^c}. \quad (3.28)$$

The last formula and (3.19) imply that

$$\begin{aligned} -\infty & < \underline{\lim}_{\varepsilon \downarrow 0} \frac{\frac{1}{2} - T_{b+\varepsilon}}{\varepsilon} \leq \overline{\lim}_{\varepsilon \downarrow 0} \frac{\frac{1}{2} - T_{b+\varepsilon}}{\varepsilon} < \infty, \\ -\infty & < \underline{\lim}_{\varepsilon \downarrow 0} \frac{\frac{1}{2} - T_{b-\varepsilon}}{\varepsilon} \leq \overline{\lim}_{\varepsilon \downarrow 0} \frac{\frac{1}{2} - T_{b-\varepsilon}}{\varepsilon} < \infty, \end{aligned} \quad (3.29)$$

and

$$\lim_{\varepsilon \downarrow 0} \frac{1 - T_{b-\varepsilon} - T_{b+\varepsilon}}{\varepsilon} = 0. \quad (3.30)$$

Therefore, from (3.22), (3.25), (3.26) and (3.27), we obtain the desired conclusion that

$$\begin{aligned}
& \lim_{\varepsilon \downarrow 0} \frac{(V(b+\varepsilon) + V(b-\varepsilon))/2 - V(b)}{\varepsilon} \\
&= \lim_{\varepsilon \downarrow 0} \left(V(b+\varepsilon) \frac{1/2 - T_{b+\varepsilon}}{\varepsilon} + V(b-\varepsilon) \frac{1/2 - T_{b-\varepsilon}}{\varepsilon} \right) \\
&= \lim_{\varepsilon \downarrow 0} \left\{ (V(b+\varepsilon) - V(b)) \frac{1/2 - T_{b+\varepsilon}}{\varepsilon} + (V(b-\varepsilon) - V(b)) \frac{1/2 - T_{b-\varepsilon}}{\varepsilon} \right\} \\
&= 0.
\end{aligned} \tag{3.31}$$

where the second equality follows from (3.30) and the third one is due to (3.29) and the result that $V(x)$ is continuous at b (see (3.8) and (3.10) and (3.12)). \square

Remark 3.1. *The number 3 appeared in the restriction of $-\vartheta < 3 \cdot q < \eta$ is not important. In fact, we use the number 3 only in the derivation of (3.23). To guarantee $\lim_{\varepsilon \downarrow 0} \mathbb{E}_b \left[e^{qX_{e(s)}} \mathbf{1}_{\{e(s) \leq \tau_{b,\varepsilon}^c\}} \right] / \varepsilon = 0$, it is enough to require that $-\vartheta < \beta \cdot q < \eta$ for some $\beta > 2$ so that $\beta_1 := \frac{\beta}{\beta-1} < 2$, which ensures that (recall (3.20))*

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}_b \left[\mathbf{1}_{\{e(s) \leq \tau_{b,\varepsilon}^c\}} \right] / \varepsilon^{\beta_1} = 0.$$

Remark 3.2. *The distribution of Y_1 in (2.1) has no influence on the derivation of (3.21), this means that formula (3.21) holds for a process X given by (1.1) and (2.1) with arbitrary jump distributions.*

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